7th Indian National Mathematical Olympiad – 1992

1. In a triangle ABC, angle A is twice angle B. Show that $a^2 = b(b + c)$.

2. If x, y and z are three real numbers such that $x + y + z = 4$ and $x^2 + y^2 + z^2 = 6$, then show that each of x, y and z lie in the closed interval $\left[\frac{2}{3}, 2\right]$, that is $\frac{2}{3} \leq x \leq 2$, $\frac{2}{3} \leq y \leq 2$, $\frac{2}{3} \leq z \leq 2$. Can x attain the extreme value $\frac{2}{3}$ or 2?

3. Find the remainder when 19^{92} is divided by 92.

4. Find the number of permutations $(p_1, p_2, p_3, p_4, p_5, p_6)$ of $1, 2, 3, 4, 5, 6$ such that for any k, $1 \leq k \leq 5$, $(p_1, p_2, \ldots, p_k)$ does not form a permutation of $1, 2, \ldots, k$. (That is, $p_1 \neq 1$, (p_1, p_2) is not a permutation of $1, 2$; (p_1, p_2, p_3) is not a permutation of $1, 2, 3$, etc.)

5. Two circles C_1 and C_2 intersect at two distinct points in a plane. Let a line passing through P meet the circles C_1 and C_2 in A and B respectively. Let Y be the midpoint of AB and let QY meet the circles C_1 and C_2 in X and Z respectively. Show that Y is also the midpoint of XZ.

6. Let $f(x)$ be a polynomial in x with integer coefficients and suppose that for five integers a_1, a_2, a_3, a_4, a_5 one has $f(a_1) = f(a_2) = f(a_3) = f(a_4) = f(a_5) = 2$. Show that there does not exist an integer b such that $f(b) = 9$.

7. Find the number of ways in which one can place the numbers $1, 2, 3, \ldots, n^2$ on the n^2 squares of an $n \times n$ chessboard, one on each, such that the numbers in each row and in each column are in arithmetic progression. Assume $n \geq 3$.

8. Determine all pairs (m, n) of positive integers for which $2^m + 3^n$ is a perfect square.

9. Let $A_1, A_2, A_3, \ldots, A_n$ be the vertices of an n-sided polygon such that
$$\frac{1}{A_1A_2} = \frac{1}{A_1A_3} + \frac{1}{A_1A_4}.$$
Determine n, the number of sides of the polygon.

10. Determine all functions $f : \mathbb{R} \setminus [0, 1] \to \mathbb{R}$ satisfying the functional relation
$$f(x) + f\left(\frac{1}{1-x}\right) = \frac{2(1-2x)}{x(1-x)},$$
where x is a real number, $x \neq 0$, $x \neq 1$.