8th Indian National Mathematical Olympiad – 1993

1. The diagonals AC and BD of a cyclic quadrilateral $ABCD$ intersect at P. Let O be the circumcenter of triangle APB and H be the orthocenter of triangle CPD. Show that the points H, P, O are collinear.

2. Let $p(x) = x^2 + ax + b$ be a quadratic polynomial in which a and b are integers. Given any integer n, show that there is an integer M such that $p(n)p(n+1) = p(M)$.

3. If a, b, c, d are four nonnegative real numbers and $a+b+c+d = 1$, show that $ab+bc+cd \leq \frac{1}{4}$.

4. Let ABC be a triangle in plane Σ. Find the set of all points P (distinct from A, B, C) in the plane Σ such that the circumcircles of triangles ABP, BCP and CAP have the same radii.

5. Show that there is a natural number n such that $n!$ when written in decimal notation (that is, in base 10) ends exactly in 1993 zeros.

6. Let ABC be a triangle right-angled at A and S be its circumcircle. Let S_1 be the circle touching the lines AB and AC, and the circle S internally. Further, let S_2 be the circle touching the lines AB and AC, and the circle S externally. If r_1 and r_2 be the radii of the circles S_1 and S_2 respectively, show that $r_1 r_2 = 4 \times area\, \Delta\, ABC$.

7. Let $A = \{1, 2, 3, \ldots, 100\}$ and B be a subset of A having 53 elements. Show that B has two distinct elements x and y whose sum is divisible by 11.

8. Let f be a bijective (one-to-one and onto) function from $A = \{1, 2, 3, \ldots, n\}$ to itself. Show that there is a positive integer $M > 1$ such that $f^M(i) = f(i)$ for each $i \in A$. (f^M denotes the composite function $f \circ f \circ \cdots \circ f$ M times.)

9. Show that there exists a convex hexagon in the plane such that:

   (a) all its interior angles are equal;

   (b) its sides are $1, 2, 3, 4, 5, 6$ in some order.