

## 9th Indian National Mathematical Olympiad – 1994

1. Let  $G$  be the centroid of the triangle  $ABC$  be a triangle in which the angle at  $C$  is obtuse and  $AD$  and  $CF$  be medians from  $A$  and  $C$  respectively onto the sides  $BC$  and  $AB$ . If the four points  $B, D, G$  and  $F$  are concyclic, show that  $\frac{AC}{BC} > \sqrt{2}$ . If further  $P$  is a point on the line  $BG$  extended such that  $AGCP$  is a parallelogram, show that the triangle  $ABC$  and  $GAP$  are similar.
2. If  $x^5 - x^3 + x = a$ , prove that  $x^6 \geq 2a - 1$ .
3. In any set of 181 square integers, prove that one can always find a subset of 19 numbers, sum of whose elements is divisible by 19.
4. Find the number of nondegenerate triangles whose vertices lie in the set of points  $(s, t)$  in the plane such that  $0 \leq s \leq 4$ ,  $0 \leq t \leq 4$ ,  $s$  and  $t$  are integers.
5. A circle passes through the vertex of a rectangle  $ABCD$  and touches its sides  $AB$  and  $AD$  at  $M$  and  $N$  respectively. If the distance from  $C$  to the line segment  $MN$  is equal to 5 units, find the area of the rectangle  $ABCD$ .
6. If  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a function satisfying the properties:
  - (a)  $f(-x) = -f(x)$ ;
  - (b)  $f(x + 1) = f(x) + 1$ ;
  - (c)  $f\left(\frac{1}{x}\right) = \frac{f(x)}{x^2}$  for  $x \neq 0$ ,

prove that  $f(x) = x$  for all real values of  $x$ . Here  $\mathbb{R}$  denotes the set of real numbers.