10th Indian National Mathematical Olympiad – 1995

1. In an acute-angled triangle ABC, $\angle A = 30^\circ$, H is the orthocentre, and M is the midpoint of BC. On the line HM take the point T such that $HM = MT$. Show that $AT = 2\,BC$.

2. Show that there are infinitely many pairs (a, b) of relatively prime integers (not necessarily positive) such that both quadratic equations

$$x^2 + ax + b = 0 \quad \text{and} \quad x^2 + 2ax + b = 0$$

   have integer roots.

3. Show that the number of 3-element subsets $\{a, b, c\}$ of $\{1, 2, 3, \ldots, 63\}$ with $a+b+c < 95$ is less than the number of those with $a + b + c \geq 95$.

4. Let ABC be a triangle and a circle Γ' be drawn lying outside the triangle, touching its incircle Γ externally, and also touching the two sides AB and C. Show that the ratio of the radii of the circles Γ' and Γ is equal to $\tan^2\left(\frac{\pi-A}{4}\right)$.

5. Let $n \geq 2$. Let $a_1, a_2, a_3, \ldots, a_n$ be n real numbers all greater than 1 and such that $|a_k - a_{k+1}| < 1$ for $1 \leq k \leq n-1$. Show that

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \frac{a_3}{a_4} + \cdots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1} < 2n - 1.$$

6. Find all primes p for which the quotient $(2^{p-1} - 1)/p$ is a square.