11th Indian National Mathematical Olympiad – 1996

1. (a) Given any positive integer n, show that there exist distinct positive integers x and y such that $x+j$ divides $y+j$ for $j=1,2,\ldots,n$.

   (b) If for some positive integers x and y, $x+j$ divides $y+j$ for all positive integers j, prove that $x=y$.

2. Let C_1 and C_2 be concentric circles in the plane with radii R and $3R$ respectively. Show that the orthocentre of any triangle inscribed in circle C_1 lies in the interior of circle C_2. Conversely, show that every point in the interior of C_2 is the orthocentre of some triangle inscribed in C_1.

3. Solve the following system of equations for real numbers a,b,c,d,e:

   $$3a=(b+c+d)^3, 3b=(c+d+e)^3, 3c=(d+e+a)^3, 3d=(e+a+b)^3, 3e=(a+b+c)^3.$$

4. Let X be a set containing n elements. Find the number of all ordered triples (A,B,C) of subsets of X such that A is a subset of B and B is a proper subset of C.

5. Define a sequence $(a_n)_{n\geq 1}$ by $a_1=1$, $a_2=2$ and

   $$a_{n+2}=2a_{n+1}-a_n+2$$

   for $n\geq 1$. Prove that for any m, $a_m a_{m+1}$ is also a term in the sequence.

6. There is a $2n\times 2n$ array (matrix) consisting of 0's and 1's and there are exactly $3n$ zeros. Show that it is possible to remove all the zeros by deleting some n rows and some n columns.
   (Note: A $m\times n$ array is a rectangular arrangement of mn numbers in which there are m horizontal rows and n vertical columns.)