13th Indian National Mathematical Olympiad – 1998

1. In a circle C_1 with centre O, let AB be a chord that is not a diameter. Let M be the midpoint of AB. Take a point T on the crcle C_2 with OM as diameter. Let the tangent to C_2 at T meet C_1 in P. Show that $PA^2 + PB^2 = 4\,PT^2$.

2. Let a and b be two positive rational numbers such that $\sqrt[3]{a} + \sqrt[3]{b}$ is also a rational number. Prove that $\sqrt[3]{a}$ and $\sqrt[3]{b}$ themselves are rational numbers.

3. Let p, q, r, s be four integers such that s is not divisible by 5. If there is an integer a such that $pa^3 + qa^2 + ra + s$ is divisible by 5, prove that there is an integer b such that $sb^3 + rb^2 + qb + p$ is also divisible by 5.

4. Suppose $ABCD$ is a cyclic quadrilateral inscribed in a circle of radius one unit. If $AB{\cdot}BC{\cdot}CD{\cdot}DA \geq 4$, prove that $ABCD$ is a square.

5. Suppose a, b, c are three real numbers such that the quadratic equation

$$x^2 - (a + b + c)x + (ab + bc + ca) = 0$$

has roots of the form $\alpha + i\beta$ where $\alpha > 0$ and $\beta \neq 0$ are real numbers, and $i^2 = -1$. Show that:

   (a) The numbers a, b, c are all positive;

   (b) The numbers $\sqrt{a}, \sqrt{b}, \sqrt{c}$ form the sides of a triangle.

6. It is desired to choose n integers from the collection of $2n$ integers $0, 0, 1, 1, 2, 2, \ldots, n-1, n-1$ such that the average (i.e., the arithmetic mean) of these n chosen integers is itself an integer and as small as possible. Show that this can be done for each positive integer n, and find this minimum value for each n.