

15th Indian National Mathematical Olympiad – 2000

1. The incircle of triangle ABC touches the sides BC, CA and AB in K, L and M respectively. The line through A and parallel to LK meets MK in P and the line through A and parallel to MK meets LK in Q . Show that the line PQ bisects the sides AB and AC of the triangle ABC .

2. Solve for integers x, y, z :

$$x + y = 1 - z, \quad x^3 + y^3 = 1 - z^2.$$

3. If a, b, c, x are real numbers such that $abc \neq 0$ and

$$\frac{xb + (1-x)c}{a} = \frac{xc + (1-x)a}{b} = \frac{xa + (1-x)b}{c},$$

then prove that $a = b = c$.

4. In a convex quadrilateral $PQRS$, $PQ = RS$, $(\sqrt{3}+1)QR = SP$ and $\angle RSP - \angle SQP = 30^\circ$. Prove that $\angle PQR - \angle QRS = 90^\circ$.
5. Let a, b, c be three real numbers such that $1 \geq a \geq b \geq c \geq 0$. Prove that if λ is a root of the cubic equation $x^3 + ax^2 + bx + c = 0$ (real or complex), then $|\lambda| \leq 1$.
6. For any natural number n , $n \geq 3$, let $f(n)$ denote the number of congruent integer-sided triangles with perimeter n (e.g., $f(3) = 1$, $f(7) = 2$). Show that:
 - (a) $f(1999) > f(1996)$;
 - (b) $f(2000) = f(1997)$.