16th Indian National Mathematical Olympiad – 2001

Time: 4 hours   **4 February 2001**

1. Let ABC be a triangle in which *no* angle is $90°$. For any point P in the plane of the triangle, let A_1, B_1, C_1 denote the reflections of P in the sides BC, CA, AB respectively. Prove the following statements:

   (a) If P is the incentre or an excentre of ABC, then P is the circumcentre of $A_1 B_1 C_1$;

   (b) If P is the circumcentre of ABC, then P is the orthocentre of $A_1 B_1 C_1$;

   (c) If P is the orthocentre of ABC, then P is either the incentre or an excentre of $A_1 B_1 C_1$.

2. Show that the equation

$$x^2 + y^2 + z^2 = (x - y)(y - z)(z - x)$$

   has infinitely many solutions in integers x, y, z.

3. If a, b, c are positive real numbers such that $abc = 1$, prove that

$$a^{b+c} b^{c+a} c^{a+b} \leq 1.$$

4. Given any nine integers show that it is possible to choose, from among them, four integers a, b, c, d such that $a + b - c - d$ is divisible by 20. Further show that such a selection is not possible if we start with eight integers instead of nine.

5. Let ABC be a triangle and D be the mid-point of side BC. Suppose $\angle DAB = \angle BCA$ and $\angle DAC = 15°$. Show that $\angle ADC$ is obtuse. Further, if O is the circumcentre of ADC, prove that triangle AOD is equilateral.

6. Let $\mathbb{R}$ denote the set of all real numbers. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the condition

$$f(x + y) = f(x)f(y)f(xy)$$

   for all x, y in $\mathbb{R}$.