17th Indian National Mathematical Olympiad – 2002

Time: 4 hours **3 February 2002**

1. For a convex hexagon $ABCDEF$, consider the following six statements:

$$\begin{array}{ll} (a_1)\ AB \text{ is parallel to } DE; & (a_2)\ AE = BD; \\ (b_1)\ BC \text{ is parallel to } EF; & (b_2)\ BF = CE; \\ (c_1)\ CD \text{ is parallel to } FA; & (c_2)\ CA = DF. \end{array}$$

   (a) Show that if all the six statements are true, then the hexagon is cyclic (i.e., it can be inscribd in a circle).

   (b) Prove that, in fact, any five of these statements also imply that the hexagon is cyclic.

2. Determine the least positive value taken by the expression $a^3 + b^3 + c^3 - 3abc$ as a, b, c vary over all positive integers. Find also all triples (a, b, c) for which this least value is attained.

3. Let x, y be positive reals such that $x + y = 2$. Prove that

$$x^3y^3(x^3 + y^3) \leq 2.$$

4. Do there exist 100 lines in the plane, no three of them concurrent, such that they intersect exactly in 2002 points?

5. Do there exist three distinct positive real numbers a, b, c such that the numbers $a, b, c, b+c-a, c+a-b, a+b-c$ and $a+b+c$ form a 7-term arithmetic progression in some order?

6. Suppose the n^2 numbers $1, 2, 3, \ldots, n^2$ are arranged to form an n by n array consisting of n rows and n columns such that the numbers in each row (from left to right) and each column (from top to bottom) are in increasing order. Denote by a_{jk} the number in the j-th row and the k-th column. Suppose b_j is the maximum possible number of entries that can occur as a_{jj}, $1 \leq j \leq n$. Prove that

$$b_1 + b_2 + b_3 + \cdots + b_n \leq \frac{n}{3}(n^2 - 3n + 5).$$

   (**Example:** In the case $n = 3$, the only numbers which can occur as a_{22} are $4, 5$ or 6 so that $b_2 = 3$.)