

18th Indian National Mathematical Olympiad – 2003

1. Consider an acute triangle ABC and let P be an interior point of ABC . Suppose the lines BP and CP , when produced, meet AC and AB in E and F respectively. Let D be the point where AP intersects the line segment EF and K be the foot of the perpendicular from D onto BC . Show that DK bisects $\angle EKF$.
2. Find all primes p and q , and even numbers $n > 2$, satisfying the equation

$$p^n + p^{n-1} + \cdots + p + 1 = q^2 + q + 1.$$

3. Show that for every real number a the equation

$$8x^4 - 16x^3 + 16x^2 - 8x + a = 0$$

has at least one non-real root, and find the sum of all the non-real roots of the equation.

4. Find all 7-digit numbers formed by using the digits 5 and 7, and divisible by both 5 and 7.
5. Let ABC be a triangle with sides a, b, c . Consider a triangle $A_1B_1C_1$ with sides equal to $a + \frac{b}{2}, b + \frac{c}{2}, c + \frac{a}{2}$. Show that

$$[A_1B_1C_1] \geq \frac{9}{4} [ABC],$$

where $[XYZ]$ denotes the area of triangle XYZ .

6. In a lottery, tickets are given nine-digit numbers using only the digits 1, 2, 3. They are also coloured red, blue or green in such a way that two tickets whose numbers differ in all the nine places get different colours. Suppose the ticket bearing the number 122222222 is red and that bearing the number 22222222 is green. Determine, with proof, the colour of the ticket bearing the number 123123123.