

20th Indian National Mathematical Olympiad – 2005

Time: 4 hours

6 February 2005

1. Let M be the midpoint of side BC of a triangle ABC . Let the median AM intersect the incircle of ABC at K and L , K being nearer A than L . If $AK = KL = LM$, prove that the sides of triangle ABC are in the ratio $5 : 10 : 13$ in some order.
2. Let α and β be positive integers such that

$$\frac{43}{197} < \frac{\alpha}{\beta} < \frac{17}{77}.$$

Find the minimum possible value of β .

3. Let p, q, r be positive real numbers, not all equal, such that some two of the equations

$$px^2 + 2qx + r = 0, \quad qx^2 + 2rx + p = 0, \quad rx^2 + 2px + q = 0,$$

have a common root, say α . Prove that

- (a) α is real and negative; and
 - (b) the remaining third equation has non-real roots.
4. All possible 6-digit numbers, in each of which the digits occur in *non-increasing* order (from left to right, e.g., 877550) are written as a sequence in *increasing* order. Find the 2005th number in this sequence.
 5. Let x_1 be a given positive integer. A sequence $\{x_n\}_{n=1}^{\infty} = \{x_1, x_2, x_3, \dots\}$ of positive integers is such that x_n , for $n \geq 2$, is obtained from x_{n-1} by adding some nonzero digit of x_{n-1} . Prove that
 - (a) the sequence has an *even* number;
 - (b) the sequence has infinitely many even numbers.
 6. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x^2 + yf(z)) = xf(x) + zf(y),$$

for all x, y, z in \mathbb{R} . (Here \mathbb{R} denotes the set of all real numbers.)