25th Indian National Mathematical Olympiad – 2010

Time: 4 hours **17 January 2010**

1. Let ABC be a triangle with circumcircle Γ. Let M be a point in the interior of the triangle ABC which is also on the bisector of $\angle A$. Let AM, BM, CM meet Γ in A_1, B_1, C_1 respectively. Suppose P is the point of intersection of A_1C_1 with AB, and Q is the point of intersection of A_1B_1 with AC. Prove that PQ is parallel to BC.

2. Find all natural numbers $n > 1$ such that n^2 does **not** divide $(n-2)!$.

3. Find all non-zero real numbers x, y, z which satisfy the system of equations:

$$\begin{aligned}(x^2+xy+y^2)(y^2+yz+z^2)(z^2+zx+x^2) &= xyz;\\ (x^4+x^2y^2+y^4)(y^4+y^2z^2+z^4)(z^4+z^2x^2+x^4) &= x^3y^3z^3.\end{aligned}$$

4. How many 6-tuples $(a_1, a_2, a_3, a_4, a_5, a_6)$ are there such that each of $a_1, a_2, a_3, a_4, a_5, a_6$ is from the set $\{1, 2, 3, 4\}$ and the six expressions

$$a_j^2 - a_j a_{j+1} + a_{j+1}^2$$

for $j = 1, 2, 3, 4, 5, 6$ (where a_7 is to be taken as a_1) are all equal to one another?

5. Let ABC be an acute-angled triangle with altitude AK. Let H be its orthocentre and O be its circumcentre. Suppose KOH is an acute-angled triangle and P its circumcentre. Let Q be the reflection of P in the line HO. Show that Q lies on the line joining the midpoints of AB and AC.

6. Define a sequence $(a_n)_{n\geq 0}$ by $a_0 = 0$, $a_1 = 1$ and

$$a_n = 2a_{n-1} + a_{n-2}$$

for $n \geq 2$.

   (a) For every $m > 0$ and $0 \leq j \leq m$, prove that $2a_m$ divides $a_{m+j} + (-1)^j a_{m-j}$.

   (b) Suppose 2^k divides n for some natural numbers n and k. Prove that 2^k divides a_n.