Regional Mathematical Olympiad – 1994

1. A leaf is torn from a paperback novel. The sum of the numbers on the remaining pages is 15000. What are the page numbers on the torn leaf?

2. In a triangle ABC, the incircle touches the sides BC, CA and AB respectively at D, E and F. If the radius of the incircle is 4 units and if BD, CE and AF are consecutive integers, find the sides of the triangle ABC.

3. Find all 6-digit natural numers $a_1a_2a_3a_4a_5a_6$ formed by using the digits $1, 2, 3, 4, 5, 6$ once each such that the number $a_1a_2 \ldots a_k$ is divisible by k for $1 \leq k \leq 6$.

4. Solve the system of equations for real x and y:

$$5x\left(1 + \frac{1}{x^2 + y^2}\right) = 12 \quad \text{and} \quad 5y\left(1 - \frac{1}{x^2 + y^2}\right) = 4.$$

5. Let A be a set of 16 positive integers with the property that the product of any two distinct members of A will not exceed 1994. Show that there are numbers a and b in the set A such that gcd of a and b is greater than 1.

6. Let AC and BD be two chords of a circle with center O such that they intersect at right-angles inside the circle at the point M. Suppose K and L are the mid-points of the chords AB and CD respectively. Prove that $OKML$ is a parallelogram.

7. Find the number of all rational numbers m/n such that:

$$0 < \frac{m}{n} < 1, \quad \gcd(m, n) = 1, \quad \text{and} \quad mn = 25!$$

8. If a, b and c are positive real numbers such that $a + b + c = 1$, prove that

$$(1 + a)(1 + b)(1 + c) \geq (1 - a)(1 - b)(1 - c).$$