Regional Mathematical Olympiad – 1995

1. In a triangle ABC, K and L are points on the side BC, K being closer to B than L, such that $BC{\cdot}KL = BK{\cdot}CL$ and AL bisects $\angle KAC$. Show that AL is perpendicular to AB.

2. Call a positive integer n good, if there are n integers, positive or negative, and not necessarily distinct, such that their sum and product are both equal to n (e.g., 8 is good, since $8 = 4{\cdot}2{\cdot}1{\cdot}1{\cdot}1{\cdot}1{\cdot}(-1){\cdot}(-1) = 4 + 2 + 1 + 1 + 1 + 1 + (-1) + (-1)$). Show that integers of the form $4k + 1$ $(k \geq 0)$ and 4ℓ $(\ell \geq 2)$ are good.

3. Prove that among any 18 consecutive 3-digit numbers there is at least one number which is divisible by the sum of its digits.

4. Show that the quadratic equation $x^2 + 7x - 14(q^2 + 1) = 0$, where q is an integer, has no integer root.

5. Show that for any triangle ABC, the following inequality is true:

$$a^2 + b^2 + c^2 > \sqrt{3}\max\{|a^2 - b^2|, |b^2 - c^2|, |c^2 - a^2|\},$$

   where a, b, c are the sides of the triangle.

6. Let $A_1A_2A_3 \ldots A_{21}$ be a 21-sided regular polygon inscribed in a circle with centre O. How many triangles $A_iA_jA_k$, $1 \leq i < j < k \leq 21$, contain the point O in their interior?

7. Show that for any real number x,

$$x^2 \sin x + x \cos x + x^2 + \frac{1}{2} > 0.$$