Regional Mathematical Olympiad – 1997

1. Let P be a interior point of a triangle ABC, and let BP and CP meet AC and AB at E and F respectively. If $[BPF] = 4$, $[BPC] = 8$ and $[CPE] = 13$, find $[AFPE]$. (Here $[\ldots]$ denotes the area of a triangle or a quadrilateral as the case may be.)

2. For each positive integer n, define $a_n = n^2 + 20$ and $d_n = \gcd(a_n, a_{n+1})$. Find the set of all values that are taken by d_n, and show by examples that each of these values is attained.

3. Solve for real x:
$$\frac{1}{[x]} + \frac{1}{[2x]} = (x) + \frac{1}{3},$$
where $[x]$ is the greatest integer less than or equal to x and $(x) = x - [x]$. (e.g., $[3.4] = 3$ and $(3.4) = 0.4$.)

4. In a quadrilateral $ABCD$, it is given that AB is parallel to CD and the diagonals AC and BD are perpendicular to each other. Show that:
   (a) $AD{\cdot}BC \geq AB{\cdot}CD$;
   (b) $AD + BC \geq AB + CD$.

5. Let x, y and z be three distinct real positive numbers. Determine, with proof, whether or not the three real numbers
$$\left|\frac{x}{y} - \frac{y}{x}\right|, \quad \left|\frac{y}{z} - \frac{z}{y}\right|, \quad \left|\frac{x}{z} - \frac{z}{x}\right|$$
can be the lengths of the sides of a triangle.

6. Find the number of unordered pairs $\{A, B\}$ (i.e., the pairs $\{A, B\}$ and $\{B, A\}$ are considered to be the same) of subsets of an n-element set X which satisfy the conditions:
   (a) $A \neq B$;
   (b) $A \cup B = X$.

   (e.g., if $X = \{a, b, c, d\}$, then $\{\{a, b\}, \{b, c, d\}\}, \{\{a\}, \{b, c, d\}\}, \{\emptyset, \{a, b, c, d\}\}$ are some of the admissible pairs.)