Regional Mathematical Olympiad – 1998

1. Let $ABCD$ be a convex quadrilateral in which $\angle BAC = 50°$, $\angle CAD = 60°$, $\angle CBD = 30°$ and $\angle BDC = 25°$. If E is the point of intersection of AC and BD, find $\angle AEB$.

2. Let n be a positive integer and $p_1, p_2, p_3, \ldots, p_n$ be n prime numbers, all greater than 5, such that 6 divides $p_1^2 + p_2^2 + p_3^2 + \cdots + p_n^2$. Prove that 6 divides n.

3. Prove for every natural number $n > 1$ the following inequality:
$$\frac{1}{n+1}\left(1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}\right) > \frac{1}{n}\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2n}\right).$$

4. Let ABC be a triangle with $AB = AC$ and $\angle BAC = 30°$. Let A' be the reflection of A in the line BC; B' be the reflection of B in the line CA; C' be the reflection of C in the line AB. Show that A', B', C' form the vertices of an equilateral triangle.

5. Find the minimum possible least common multiple (lcm) of twenty (not necessarily distinct) natural numbers whose sum is 801.

6. Given the 7-element set $A = \{a, b, c, d, e, f, g\}$, find a collection T of 3-element subsets of A such that each pair of elements from A occurs exactly in one of the subsets of T.