Regional Mathematical Olympiad – 2001

Time: 3 hours **2 December 2001**

1. Let BE and CF be the altitudes of an acute triangle ABC, with E on AC and F on AB. Let O be the point of intersection of BE and CF. Take any line KL through O with K on AB and L on AC. Suppose M and N are located on BE and CF respectively, such that KM is perpendicular to BE and LN is perpendicular to CF. Prove that FM is parallel to EN.

2. Find all primes p and q such that $p^2 + 7pq + q^2$ is the square of an integer.

3. Find the number of positive integers x which satisfy the condition

$$\left[\frac{x}{99}\right] = \left[\frac{x}{101}\right].$$

(Here $[z]$ denotes, for any real z, the largest integer not exceeding z; e.g. $[7/4] = 1$.)

4. Consider an $n \times n$ array of numbers:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}$$

Suppose each row consists of the n numbers $1, 2, 3, \ldots, n$ in some order and $a_{ij} = a_{ji}$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$. If n is odd, prove that the numbers $a_{11}, a_{22}, a_{33}, \ldots, a_{nn}$ are $1, 2, 3, \ldots, n$ in some order.

5. In a triangle ABC, D is a point on BC such that AD is the internal bisector of $\angle A$. Suppose $\angle B = 2\angle C$ and $CD = AB$. Prove that $\angle A = 72^\circ$.

6. If x, y, z are the sides of a triangle. then prove that

$$|\, x^2(y - z) + y^2(z - x) + z^2(x - y)\,| < xyz.$$

7. Prove that the product of the first 1000 positive even integers differs from the product of the first 1000 positive odd integers by a multiple of 2001.