Regional Mathematical Olympiad – 2002

1. In an acute triangle ABC, points D, E, F are located on the sides BC, CA, AB respectively such that
$$\frac{CD}{CE} = \frac{CA}{CB}, \quad \frac{AE}{AF} = \frac{AB}{AC}, \quad \frac{BF}{BD} = \frac{BC}{BA}.$$
Prove that AD, BE, CF are the altitudes of ABC.

2. Solve the following equation for real x:
$$(x^2 + x - 2)^3 + (2x^2 - x - 1)^3 = 27(x^2 - 1)^3.$$

3. Let a, b, c be positive integers such that a divides b^2, b divides c^2 and c divides a^2. Prove that abc divides $(a + b + c)^7$.

4. Suppose the integers $1, 2, 3, \ldots, 10$ are split into two disjoint collections a_1, a_2, a_3, a_4, a_5 and b_1, b_2, b_3, b_4, b_5 such that $a_1 < a_2 < a_3 < a_4 < a_5$ and $b_1 < b_2 < b_3 < b_4 < b_5$.
   (i) Show that the larger number in any pair $\{a_j, b_j\}$, $1 \le j \le 5$, is at least 6.
   (ii) Show that $|a_1 - b_1| + |a_2 - b_2| + |a_3 - b_3| + |a_4 - b_4| + |a_5 - b_5| = 25$ for every such partition.

5. The circumference of a circle is divided into eight equal arcs by a convex quadrilateral $ABCD$, with four arcs lying inside the quadrilateral and the remaining four lying outside it. The lengths of the arcs lying inside the quadrilateral are denoted by p, q, r, s in counter-clockwise direction starting from some arc. Suppose $p + r = q + s$. Prove that $ABCD$ is a cyclic quadrilateral.

6. For any natural number $n > 1$, prove the inequality:
$$\frac{1}{2} < \frac{1}{n^2+1} + \frac{2}{n^2+2} + \frac{3}{n^2+3} + \cdots + \frac{n}{n^2+n} < \frac{1}{2} + \frac{1}{2n}.$$

7. Find all integers a, b, c, d satisfying the following relations:
   (i) $1 \le a \le b \le c \le d$;
   (ii) $ab + cd = a + b + c + d + 3$.