Regional Mathematical Olympiad – 2005

Time: 3 hours **4 December 2005**

1. Let $ABCD$ be a convex quadrilateral; P, Q, R, S be the midpoints of AB, BC, CD, DA respectively such that the triangles AQR and CSP are equilateral. Prove that $ABCD$ is a rhombus. Determine its angles.

2. If x, y are integers, and 17 divides both the expressions $x^2 - 2xy + y^2 - 5x + 7y$ and $x^2 - 3xy + 2y^2 + x - y$, then prove that 17 divides $xy - 12x + 15y$.

3. If a, b, c are three real numbers such that $|a - b| \geq |c|$, $|b - c| \geq |a|$, $|c - a| \geq |b|$, then prove that one of a, b, c is the sum of the other two.

4. Find the number of all 5-digit numbers (in base 10) each of which contains the block 15 and is divisible by 15. For example, 31545, 34155 are two such numbers.

5. In a triangle ABC, let D be the midpoint of BC. If $\angle ADB = 45°$ and $\angle ACD = 30°$, determine $\angle BAD$.

6. Determine all triples (a, b, c) of positive integers such that $a \leq b \leq c$ and

$$a + b + c + ab + bc + ca = abc + 1.$$

7. Let a, b, c be three positive real numbers such that $a + b + c = 1$. Let

$$\lambda = \min\{a^3 + a^2bc, b^3 + ab^2c, c^3 + abc^2\}.$$

Prove that the roots of the equation $x^2 + x + 4\lambda = 0$ are real.