

# Regional Mathematical Olympiad – 2006

Time: 3 hours

3 December 2006

1. Let  $ABC$  be an acute-angled triangle and let  $D, E, F$  be the feet of perpendiculars from  $A, B, C$  respectively to  $BC, CA, AB$ . Let the perpendiculars from  $F$  to  $CB, CA, AD, BE$  meet them in  $P, Q, M, N$  respectively. Prove that  $P, Q, M, N$  are *collinear*.
2. Find the *least* possible value of  $a + b$ , where  $a, b$  are positive integers such that 11 divides  $a + 13b$  and 13 divides  $a + 11b$ .
3. If  $a, b, c$  are three positive real numbers, prove that

$$\frac{a^2 + 1}{b + c} + \frac{b^2 + 1}{c + a} + \frac{c^2 + 1}{a + b} \geq 3.$$

4. A  $6 \times 6$  squares is dissected into 9 rectangles by lines parallel to its sides such that all these rectangles have only integer sides. Prove that there are always **two** congruent rectangles.
5. Let  $ABCD$  be a quadrilateral in which  $AB$  is parallel to  $CD$  and perpendicular to  $AD$ ;  $AB = 3CD$ ; and the area of the quadrilateral is 4. If a circle can be drawn touching all the sides of the quadrilateral, find its radius.
6. Prove that there are infinitely many positive integers  $n$  such that  $n(n + 1)$  can be expressed as a sum of two positive squares in *at least* two different ways. (Here  $a^2 + b^2$  and  $b^2 + a^2$  are considered as the same representation.)
7. Let  $X$  be the set of all positive integers greater than or equal to 8 and let  $f : X \rightarrow X$  be a function such that  $f(x + y) = f(xy)$  for all  $x \geq 4, y \geq 4$ . If  $f(8) = 9$ , determine  $f(9)$ .