Regional Mathematical Olympiad – 2010

Time: 3 hours **5 December 2010**

1. Let $ABCDEF$ be a convex hexagon in which the diagonals AD, BE, CF are concurrent at O. Suppose the area of triangle OAF is the geometric mean of those of OAB and OEF; and the area of the triangle OBC is the geometric mean of those of OAB and OCD. Prove that the area of triangle OED is the geometric mean of those of OCD and OEF.

2. Let $P_1(x) = ax^2 - bx - c,\ P_2(x) = bx^2 - cx - a,\ P_3(x) = cx^2 - ax - b$ be three quadratic polynomials where a, b, c are non-zero real numbers. Suppose there exists a real number α such that $P_1(\alpha) = P_2(\alpha) = P_3(\alpha)$. Prove that $a = b = c$.

3. Find the number of 4-digit numbers (in base 10) having non-zero digits and which are divisible by 4 but not by 8.

4. Find three distinct positive integers with the least possible sum such that the sum of the reciprocals of any two integers among them is an integral multiple of the reciprocal of the third integer.

5. Let ABC be a triangle in which $\angle A = 60^\circ$. Let BE and CF be the bisectors of the angles $\angle B$ and $\angle C$ with E on AC and F on AB. Let M be the reflection of A in the line EF. Prove that M lies on BC.

6. For each integer $n \geq 1$, define $a_n = \left[\frac{n}{[\sqrt{n}]}\right]$, where $[x]$ denotes the largest integer not exceeding x, for any real number x. Find the number of all n in the set $\{1, 2, 3, \ldots, 2010\}$ for which $a_n > a_{n+1}$.