Regional Mathematical Olympiad – 2011

Time: 3 hours **4 December 2011**

1. Let ABC be a triangle. Let D, E, F be points respectively on the segments BC, CA, AB such that AD, BE, CF concur at the point K. Suppose $BD/DC = BF/FA$ and $\angle ADB = \angle AFC$. Prove that $\angle ABE = \angle CAD$.

2. Let $(a_1, a_2, a_3, \ldots, a_{2011})$ be a permutation (that is, a rearrangement) of the numbers $1, 2, 3, \ldots, 2011$. Show that there exist two numbers j, k such that $1 \leq j < k \leq 2011$ and $|a_j - j| = |a_k - k|$.

3. A natural number n is chosen strictly between two consecutive perfect squares. the smaller of these two squares is obtained by subtracting k from n and the larger one is obtained by adding ℓ to n. Prove that $n = k\ell$ is a perfect square.

4. Consider a 20-sided convex polygon K, with vertices $A_1, A_2, \ldots, A_{20}$ in that order. Find the number of ways in which three sides of K can be chosen so that every pair among them has at least two sides of K between them. (For example $(A_1A_2, A_4A_5, A_{11}A_{12})$ is an admissible triple while $(A_1A_2, A_4A_5, A_{19}A_{20})$ is not.)

5. Let ABC be a triangle and let BB_1, CC_1 be respectively the bisectors of $\angle B, \angle C$, with B_1 on AC and C_1 on AB. Let E, F be the feet of the perpendiculars drawn from A onto BB_1, CC_1 respectively. Suppose D is the point at which the incircle of ABC touches AB. Prove that $AD = EF$.

6. Find all pairs (x, y) of real numbers such that

$$16^{x^2+y} + 16^{x+y^2} = 1.$$